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The Bihari Inequality

with Applications

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Craiova, May 18, 2015 *O.G.M.*

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Preface

A simple integral inequality, due to I. Bihari, can be used profitably to compute the asymptotic behavior of solutions to nonlinear ordinary differential equations.

Craiova, [October 15, 2015] *O.G.M.*

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Acronyms

ODE Ordinary differential equation

Chapter 1
Bihari's inequality

1.1 The direct inequality

Assume that $u, a : [t_0, T) \to [0, +\infty)$, where $t_0 \geq 1$, $T \leq +\infty$, and $g : [0, +\infty) \to [0, +\infty)$ are continuous functions, g being also non-decreasing and such that $g(x) > 0$ for all $x \geq k \geq 0$.

We claim that *if there exist $c \geq k$ such that*

$$u(t) \leq c + \int_{t_0}^{t} a(s)g(u(s))ds, \quad t \in [t_0, T), \tag{1.1}$$

and respectively $c < C \leq +\infty$ such that

$$\int_{t_0}^{T} a(s)ds < \int_{c}^{C} \frac{du}{g(u)}, \tag{1.2}$$

then

$$u(t) \leq G^{-1}\left(G(c) + \int_{t_0}^{t} a(s)ds\right) < C, \quad t \in [t_0, T), \tag{1.3}$$

where

$$G(x) = \int_{k}^{x} \frac{du}{g(u)}, \quad x \geq k. \tag{1.4}$$

To prove this claim, introduce the quantity — nothing but the right hand member of (1.1) —

$$y(t) = c + \int_{t_0}^{t} a(s)g(u(s))ds, \quad t \in [t_0, T),$$

and recast (1.1) as

$$u(t) \leq y(t), \quad t \in [t_0, T). \tag{1.5}$$

Notice that $y(t_0) = c$ and $y(t) \geq c \geq k$. Consequently, $g(y(t)) > 0$ in $[t_0, T)$. We have

$$y'(t) = a(t)g(u(t)) \leq a(t)g(y(t))$$

and

$$\frac{y'(t)}{g(y(t))} \leq a(t), \quad t \in [t_0, T).$$

By integration, we deduce that

$$G(y(t)) - G(c) = \int_c^{y(t)} \frac{du}{g(u)} = \int_{t_0}^t \frac{y'(s)}{g(y(s))} ds \tag{1.6}$$

$$\leq \int_{t_0}^t a(s)ds. \tag{1.7}$$

Observe now that — recall (1.2) —

$$G(c) + \int_{t_0}^t a(s)ds < G(c) + \int_c^C \frac{du}{g(u)} = G(C), \quad t \in [t_0, T). \tag{1.8}$$

The function $G : [k, C] \to [0, G(C)]$ is bijective and increasing. We tacitly assume that C is bounded. The case when $C = +\infty$ can be handled almost verbatim.

Further, the function y being continuous and non-decreasing, and $y(t_0) = c \in [k, C)$, it is obvious that $y(t) \in [k, C)$ on a certain interval I to the right of t_0. By taking into account (1.7), (1.8), we deduce that $[t_0, T) \subseteq I$. Moreover,

$$y(t) \leq G^{-1}\left(G(c) + \int_{t_0}^t a(s)ds\right) < C, \quad t \in [t_0, T).$$

This follows from the monotonicity of G^{-1}.

The estimate (1.3) is a consequence of (1.5). The validity of our claim is now established.

The *Bihari inequality* (1.1), (1.3) was presented first in [2, pp. 83–85]. The case of reverse sign in (1.1) is discussed in [2, pp. 92–93]. The cases when $C = +\infty$ and the right hand member of (1.2) is divergent are discussed in [3, p. 262].

A similar inequality, namely

$$u(t) \leq b(t)\left[c + \int_{t_0}^t a(s)g(u(s))ds\right], \quad t \in [t_0, T),$$

has been studied by M. Golomb [5, p. 273, Eq. (2.6)] when the continuous nonlinearity g is assumed to be non-decreasing, with $g(x) > 0$ for every $x > 0$, and also sub-multiplicative. The latter property yields the key estimate regarding G given below — here, $c > 0$ and $k = 1$ —

$$G(u+v)-G(u) \geq \frac{u}{g(u)} \cdot G\left(1+\frac{v}{u}\right), \quad u>0,\ v \geq 0,$$

see [5, p. 274]. This inequality is used in conjunction with (1.6) to prove that

$$u(t) \leq b(t) \cdot c\, G^{-1}\left(\frac{g(c)}{c} \cdot \int_{t_0}^{t} a(s) g(b(s)) ds\right), \quad t \in [t_0, T).$$

The case of a sub-additive nonlinearity g is undertaken in [7, p. 491]. For other developments, see [4, 11, 13].

We shall end this subsection with a remark concerning the middle term in the double inequality (1.3). For any $\alpha \in [0, C-k)$, introduce the quantity

$$G_\alpha(x) = \int_{k+\alpha}^{x} \frac{du}{g(u)}, \quad x \in [k, C).$$

Obviously, the function $G_\alpha : [k, C) \to \left[-\int_k^{k+\alpha} \frac{du}{g(u)}, \int_{k+\alpha}^{C} \frac{du}{g(u)}\right)$ is bijective and increasing.

We notice also that, given $\alpha, \beta \in [0, C-k)$,

$$G_\beta(x) = G_\alpha(x) + q, \quad q = \int_{k+\beta}^{k+\alpha} \frac{du}{g(u)}, \tag{1.9}$$

and respectively

$$x = G_\beta^{-1}(G_\beta(x)) = G_\beta^{-1}(G_\alpha(x)+q) = G_\alpha^{-1}(G_\alpha(x)), \quad x \in [k, C).$$

The latter identity reads as

$$G_\beta^{-1}(y+q) = G_\alpha^{-1}(y), \quad y \in G_\alpha([k, C)). \tag{1.10}$$

Now, by means of (1.9), (1.10), we get — see also [2, p. 84] —

$$\begin{aligned} G_\alpha^{-1}\left(G_\alpha(c) + \int_{t_0}^{t} a(s) ds\right) &= G_\alpha^{-1}\left(G_\beta(c) - q + \int_{t_0}^{t} a(s) ds\right) \\ &= G_\beta^{-1}\left(\left[G_\beta(c) - q + \int_{t_0}^{t} a(s) ds\right] + q\right) \\ &= G_\beta^{-1}\left(G_\beta(c) + \int_{t_0}^{t} a(s) ds\right). \end{aligned} \tag{1.11}$$

The remark based on the previous computations is that, *given the relations (1.1), (1.4), there is no need to ask that $c \geq k$, the only restriction being given by $c, k >$* $\sup\{x \geq 0 : g(x) = 0\}$.

1.2 The dual inequality. Ouyang's case

Let the function $g : [0,+\infty) \to [0,+\infty)$ be continuous, bijective and increasing (of course, these conditions yield $g(x) > 0$ for all $x > 0$). Assume also that

$$\int_{0+}^{1} \frac{du}{g^{-1}(u)} < +\infty$$

and introduce $H : [0,+\infty) \to [0,+\infty)$ given by the formulas

$$H(0) = 0 \quad \text{and} \quad H(x) = \int_{0+}^{x} \frac{du}{g^{-1}(u)},\ x > 0.$$

Introduce $0 < c < C \leq +\infty$, $T \leq +\infty$ and the continuous function $a : [t_0, T) \to [0,+\infty)$ such that

$$\int_{t_0}^{T} a(s)ds < \int_{c}^{C} \frac{du}{g(u)}.$$

We claim that, given the continuous function $u : [t_0, T) \to [0,+\infty)$, *(i) if*

$$g(u(t)) \leq c + \int_{t_0}^{t} a(s)u(s)ds, \quad t \in [t_0, T), \tag{1.12}$$

then we have

$$u(t) \leq (H \circ g)^{-1} \left(H(c) + \int_{t_0}^{t} a(s)ds \right), \quad t \in [t_0, T); \tag{1.13}$$

(ii) if

$$g(u(t)) \geq c + \int_{t_0}^{t} a(s)u(s)ds, \quad t \in [t_0, T),$$

then we have

$$u(t) \geq (H \circ g)^{-1} \left(H(c) + \int_{t_0}^{t} a(s)ds \right), \quad t \in [t_0, T).$$

We shall prove only part (i). Then, for $y(t) = \int_{t_0}^{t} a(s)u(s)ds$, with $t \in [t_0, T)$, the inequality (1.12) reads as

$$u(t) \leq g^{-1}(c + y(t)). \tag{1.14}$$

Notice also that $y(t_0) = 0$.

We deduce that

$$\frac{(c+y)'(t)}{g^{-1}(c+y(t))} \leq a(t), \quad t \in [t_0, T),$$

and further, by integration, we get

$$\int_c^{c+y(t)} \frac{du}{g^{-1}(u)} \leq \int_{t_0}^t a(s)ds.$$

The latter estimate is recast as

$$H(c+y(t)) \leq H(c) + \int_{t_0}^t a(s)ds < H(C), \quad t \in [t_0, T).$$

In particular,

$$c+y(t) \leq H^{-1}\left(H(c) + \int_{t_0}^t a(s)ds\right). \tag{1.15}$$

It is obvious now that the estimate (1.13) is a consequence of the inequalities (1.14), (1.15).

The following particular case of the *dual Bihari inequality* (1.12), (1.13) is used quite frequently in stability theory and it has been discovered by L. Ouyang [10].

If $x, A : [t_0, T) \to \mathbb{R}$ *are continuous functions such that*

$$\frac{[x(t)]^2}{2} \leq \frac{[x(t_0)]^2}{2} + \int_{t_0}^t A(s)x(s)ds, \quad t \in [t_0, T),$$

then

$$|x(t)| \leq |x(t_0)| + \int_{t_0}^t |A(s)|ds, \quad t \in [t_0, T).$$

To establish Ouyang's inequality, it is enough to take

$$g(x) = \frac{x^2}{2} = H^{-1}(x), \quad g^{-1}(x) = \sqrt{2x} = H(x)$$

and

$$u(t) = |x(t)|, \quad a(t) = |A(t)|, \quad c = |u(t_0)|$$

in (1.12).

We shall recall in the end of this subsection certain connections between g and g^{-1}.

Let $g : [a, +\infty) \to [b, +\infty)$ be a bijective, increasing C^1–function. Here, $a, b > 0$ — obviously, $g(a) = b$ —.

Then,

$$\int_a^x \frac{du}{g(u)} = -\frac{a}{g(a)} + \frac{x}{g(x)} + \int_b^{g(x)} \frac{g^{-1}(u)}{u^2}du, \quad x \geq a, \tag{1.16}$$

and

$$\int_b^y \frac{du}{g^{-1}(u)} = -\frac{b}{g^{-1}(b)} + \frac{y}{g^{-1}(y)} + \int_a^{g^{-1}(y)} \frac{g(v)}{v^2} dv, \quad y \geq b. \tag{1.17}$$

The formulas (1.16), (1.17) can be established easily by differentiation with respect to x and respectively y in both sides. Also, the smoothness of g can be reduced to merely continuity if we use the Riemann-Stieltjes integration, see, for instance, [12, p. 104].

Going further, let us show that $\int_a^{+\infty} \frac{du}{g(u)} < +\infty$ yields $\int_b^{+\infty} \frac{dv}{g^{-1}(v)} = +\infty$ and $\int_b^{+\infty} \frac{dv}{g^{-1}(v)} < +\infty$ yields $\int_a^{+\infty} \frac{du}{g(u)} = +\infty$.

In fact, if $\int_a^{+\infty} \frac{du}{g(u)} < +\infty$ then we can use the following classical trick, see [6, p. 72],

$$\begin{aligned} \frac{x}{g(x)} &= \frac{1}{\ln 2} \cdot \frac{x}{g(x)} \cdot \int_{\frac{x}{2}}^{x} \frac{du}{u} = \frac{1}{\ln 2} \cdot \frac{\left[2 \cdot \frac{x}{2}\right]}{g(x)} \cdot \int_{\frac{x}{2}}^{x} \frac{du}{u} \\ &\leq \frac{2}{\ln 2} \cdot \int_{\frac{x}{2}}^{x} \frac{u}{g(u)} \cdot \frac{du}{u} \\ &\leq \frac{2}{\ln 2} \cdot \int_{\frac{x}{2}}^{+\infty} \frac{du}{g(u)}. \end{aligned} \tag{1.18}$$

Now, since $\lim\limits_{x\to+\infty} \int_{\frac{x}{2}}^{+\infty} \frac{du}{g(u)} = 0$, the estimate (1.18) yields

$$\lim_{x\to+\infty} \frac{x}{g(x)} = 0.$$

This formula can be recast as $\lim\limits_{y\to+\infty} \frac{g^{-1}(y)}{y} = 0$. Consequently,

$$\lim_{y\to+\infty} \frac{y}{g^{-1}(y)} = +\infty. \tag{1.19}$$

Since, by means of (1.18),

$$\frac{y}{g^{-1}(y)} \leq \frac{2}{\ln 2} \cdot \int_{\frac{y}{2}}^{+\infty} \frac{dv}{g^{-1}(v)} \leq +\infty,$$

the estimate (1.19) leads to $\int_b^{+\infty} \frac{dv}{g^{-1}(v)} = +\infty$.

A different proof of (1.19), based on Young's inequality, can be adapted from the computations done in [12, p. 104]. The other claim is established similarly.

Chapter 2
Bihari-like results of asymptotic integration

The simple differentiation-integration procedure that constitutes the core of Bihari's estimates (1.3), (1.13) is used next to study the long-time behavior of solutions to certain nonlinear ordinary differential equations in the real field.

2.1 First order ODE's

Consider the nonlinear ordinary differential equation

$$x' = a(t)w\left(\frac{x}{t}\right), \qquad t \geq t_0 \geq 1, \tag{2.1}$$

where $a : [t_0, +\infty) \to (0, +\infty)$, $w : \mathbb{R} \to \mathbb{R}$ are continuous and w satisfies the sign condition

$$u \cdot w(u) > 0 \quad \text{for all } u \neq 0.$$

Furthermore, assume that the restriction of w to $(0, +\infty)$ is non-decreasing. It is obvious that, since 0 is the only zero of w, we must have

$$w(u) > 0 \quad \text{for all } u > 0. \tag{2.2}$$

Given the numbers $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, introduce the quantities

$$b(t) = [a(t)]^p, \quad [d(T)]^{\frac{1}{q}} = \sup_{t \in [t_0, T]} \left(\frac{\int_{t_0}^t b(s)ds}{t^p} \right)^{\frac{1}{p}}, \qquad T \geq t \geq t_0.$$

Introduce also the quantities

$$c > 0, \quad \mathscr{W}(u) = \int_0^u \frac{d\xi}{\left[w\left(c+\xi^{\frac{1}{q}}\right)\right]^q},\ u \geq 0$$

and suppose that

$$\lim_{u\to+\infty} \frac{\mathscr{W}(u)}{u^\alpha} = +\infty \tag{2.3}$$

for a certain $\alpha \in (0,1)$.

The connection between a and w is given by the restriction

$$\lim_{T\to+\infty} T \cdot d(T) = +\infty. \tag{2.4}$$

We claim that *all the continuable solutions* $x(t)$ *with* $x(t_0) = x_0$ *and* $c = \frac{x_0}{t_0}$ *have the asymptotic behavior given by*

$$\lim_{t\to+\infty} \frac{x(t)}{t} \cdot [td(t)]^{-\frac{1}{\alpha q}} = 0. \tag{2.5}$$

Since $x(t_0) > 0$ and $x'(t_0) = a(t_0)w(c) > 0$, it is easy to see that any solution $x(t)$ of equation (2.1) starting from $x(t_0) = x_0$ is positive and increasing throughout $[t_0, +\infty)$.

The integral identity

$$x(t) = x_0 + \int_{t_0}^t a(s)w\left(\frac{x(s)}{s}\right)ds, \quad t \geq t_0,$$

is recast as

$$y(t) = \int_{t_0}^t a(s)w\left(\frac{x_0+y(s)}{s}\right)ds, \quad \text{where } y(t) = x(t) - x_0,\ t \geq t_0.$$

Notice that $y(t) > 0$ for $t > t_0$.

An application of Hölder's inequality in integral form leads to

$$y(t) \leq \left(\int_{t_0}^t [a(s)]^p ds\right)^{\frac{1}{p}} \cdot \left(\int_{t_0}^t \left[w\left(c+\frac{y(s)}{s}\right)\right]^q ds\right)^{\frac{1}{q}}$$

and respectively to

$$\frac{y(t)}{t} \leq \left(\frac{\int_{t_0}^t b(s)ds}{t^p}\right)^{\frac{1}{p}} \cdot [z(t)]^{\frac{1}{q}}, \quad z(t) = \int_{t_0}^t \left[w\left(c+\frac{y(s)}{s}\right)\right]^q ds.$$

We get

$$\frac{y(t)}{t} \leq [d(T)\cdot z(t)]^{\frac{1}{q}}, \qquad t \in [t_0, T],\ T < +\infty. \tag{2.6}$$

We have next that

$$z'(t) = \left[w\left(c+\frac{y(t)}{t}\right)\right]^q \leq \left[w\left(c+[d(T)\cdot z(t)]^{\frac{1}{q}}\right)\right]^q, \qquad t\in[t_0,T],$$

by means of (2.6).

Now,

$$\frac{[d(T)z]'(t)}{\left[w\left(c+[d(T)z(t)]^{\frac{1}{q}}\right)\right]^q} \leq d(T).$$

An integration over $[t_0,t]$, where $t\leq T$, yields — notice that $z(t_0)=0$ and recall (2.4) —

$$\begin{aligned}\mathscr{W}(d(T)z(t)) - \mathscr{W}(0) &= \int_{t_0}^{t} \frac{[d(T)z]'(s)}{\left[w\left(c+[d(T)z(s)]^{\frac{1}{q}}\right)\right]^q} ds \\ &\leq d(T)(t-t_0) \leq d(T)T\end{aligned}$$

for every $t_0\leq t\leq T<+\infty$.

We deduce further, via (2.6), that

$$\mathscr{W}\left(\left[\frac{y(T)}{T}\right]^q\right) \leq \mathscr{W}(d(T)z(T)) \leq d(T)T,$$

respectively that

$$\frac{\mathscr{W}\left(\left[\frac{y(T)}{T}\right]^q\right)}{\left[\frac{y(T)}{T}\right]^{\alpha q}} \cdot \left\{\frac{y(T)}{T[d(T)T]^{\frac{1}{\alpha q}}}\right\}^{\alpha q} \leq 1, \quad T\geq t_0.$$

If we suppose, for the sake of contradiction, that (2.5) doesn't hold true then there exist $\varepsilon>0$ and an increasing, unbounded from above, sequence $(T_n)_{n\geq 1}$ taken from $(t_0,+\infty)$ such that

$$\frac{y(T_n)}{T_n[d(T_n)T_n]^{\frac{1}{\alpha q}}} \geq \varepsilon, \quad n\geq 1.$$

This estimate can be recast as

$$\frac{y(T_n)}{T_n} \geq \varepsilon\cdot[d(T_n)T_n]^{\frac{1}{\alpha q}}. \tag{2.7}$$

By taking into account (2.4), (2.7), we obtain that

$$\lim_{n\to+\infty}\frac{y(T_n)}{T_n} = +\infty.$$

In conclusion, since

$$\frac{\mathscr{W}\left(\left[\frac{y(T_n)}{T_n}\right]^q\right)}{\left[\frac{y(T_n)}{T_n}\right]^{\alpha q}} \cdot \varepsilon^{\alpha q} \leq 1, \quad n \geq 1,$$

we arrive at a contradiction — recall (2.3) —.

The hypotheses which lead to the estimate (2.5) being complicate, it is natural to wonder about their efficiency. The following example will show that *the asymptotic behavior described by (2.5) is not raw*.

Set the numbers $\zeta > 0$, $\alpha \in (0,1)$ and $\varepsilon \in \left(0, \frac{1-\alpha}{\alpha}\right)$. Consider the equation

$$x' = t^{\zeta+(1+\varepsilon)\alpha-1} \cdot x^{1-(1+\varepsilon)\alpha}, \quad t \geq t_0 \geq 1. \tag{2.8}$$

It is obvious that $1 > (1+\varepsilon)\alpha$. We have also

$$a(t) = t^{\zeta}, \quad w(u) = u^{1-(1+\varepsilon)\alpha} \text{ for every } u \geq 0.$$

Fix $p > 1$ such that — if $\zeta \in (0,1)$ — $(1-\zeta)p \leq 1$ and introduce $q = \frac{p}{p-1}$. To verify the hypothesis (2.3), introduce $u_c = c^q$. Now,

$$\int_0^u \frac{d\xi}{\left[w\left(c+\xi^{\frac{1}{q}}\right)\right]^q} \geq \int_{u_c}^u \frac{d\xi}{\left[w\left(2\cdot\xi^{\frac{1}{q}}\right)\right]^q} = 2^{-q[1-(1+\varepsilon)\alpha]} \cdot \int_{u_c}^u \frac{d\xi}{\xi^{1-(1+\varepsilon)\alpha}}$$
$$= \frac{2^{-q[1-(1+\varepsilon)\alpha]}}{(1+\varepsilon)\alpha} \cdot \left(u^{(1+\varepsilon)\alpha} - u_c^{(1+\varepsilon)\alpha}\right) = +\infty \cdot u^{\alpha} \quad \text{when } u \to +\infty.$$

To check the hypothesis (2.4), let us start by observing that

$$\frac{d}{dt}\left[\frac{\int_{t_0}^t b(s)ds}{t^p}\right] = \frac{1}{t^p}\left[b(t) - \frac{p}{t}\int_{t_0}^t b(s)ds\right]$$
$$= t^{-p}\left(t^{\zeta p} - \frac{p}{1+\zeta p} \cdot \frac{t^{1+\zeta p} - t_0^{1+\zeta p}}{t}\right)$$
$$> t^{-p}\left(t^{\zeta p} - \frac{p}{1+\zeta p} \cdot t^{\zeta p}\right)$$
$$= \frac{1-(1-\zeta)p}{1+\zeta p} \cdot t^{(\zeta-1)p} > 0, \quad t \geq t_0.$$

So, we get

$$d(T) = \left(\frac{\int_{t_0}^T b(s)ds}{T^p}\right)^{\frac{q}{p}} = \left[\frac{T^{1+\zeta p} - t_0^{1+\zeta p}}{(1+\zeta p) \cdot T^p}\right]^{\frac{1}{p-1}}$$
$$\sim \text{constant} \cdot T^{\zeta \cdot \frac{p}{p-1} - 1} \quad \text{when } T \to +\infty$$

and respectively

$$T \cdot d(T) \sim \text{constant} \cdot T^{\zeta \cdot \frac{p}{p-1}} \quad \text{when } T \to +\infty. \tag{2.9}$$

The estimate (2.5) implies, by means of (2.9), that

$$\frac{x(t)}{t} = o\left(\left(t^{\zeta \cdot \frac{p}{p-1}}\right)^{\frac{1}{\alpha q}}\right) = o\left(t^{\frac{\zeta}{\alpha}}\right) \quad \text{as } t \to +\infty. \tag{2.10}$$

On the other hand, by taking $x_0 > 0$, a direct computation of the solution to (2.8) yields

$$\frac{[x(t)]^{(1+\varepsilon)\alpha} - x_0^{(1+\varepsilon)\alpha}}{(1+\varepsilon)\alpha} = \frac{t^{\zeta+(1+\varepsilon)\alpha} - t_0^{\zeta+(1+\varepsilon)\alpha}}{\zeta+(1+\varepsilon)\alpha}$$

and respectively

$$\begin{aligned} x(t) &\sim \text{constant} \cdot t^{\frac{\zeta+(1+\varepsilon)\alpha}{(1+\varepsilon)\alpha}} \\ &= \text{constant} \cdot t^{1+\frac{\zeta}{(1+\varepsilon)\alpha}} \quad \text{when } t \to +\infty. \end{aligned}$$

This computation shows that

$$\frac{x(t)}{t} = O\left(t^{\frac{1}{1+\varepsilon} \cdot \frac{\zeta}{\alpha}}\right) = o\left(t^{\frac{\zeta}{\alpha}}\right) \quad \text{when } t \to +\infty. \tag{2.11}$$

It is clear now that (2.11) is "infinitesimally close" to (2.10) when $\varepsilon \sim 0$.

2.2 Second order ODE's

Consider the nonlinear ordinary differential equation

$$x'' = a(t)w\left(\frac{x}{t}\right), \quad t \geq t_0 \geq 1, \tag{2.12}$$

where the continuous coefficient $a : [t_0, +\infty) \to \mathbb{R}$ verifies the restriction

$$\int_{t_0}^{+\infty} |a(t)| dt < +\infty \tag{2.13}$$

and the nonlinearity w is subjected to the same restrictions as in the preceding subsection.

Taking into account (2.2), we shall discuss the following two cases:

$$(i) \int_{0+}^{1} \frac{du}{w(u)} = +\infty \quad \text{and} \quad (ii) \int_{1}^{+\infty} \frac{du}{w(u)} = +\infty. \tag{2.14}$$

The quantity 1 from the integrals in (2.14) can be replaced conveniently with any other positive number — recall the discussion regarding (1.11) —.

We claim that *in the case (ii), all the positive continuable solutions x of (2.12) are described asymptotically by*

$$\lim_{t\to+\infty} x'(t) = \lim_{t\to+\infty} \frac{x(t)}{t} = l_x \in [0,+\infty) \tag{2.15}$$

and there exist solutions with $l_x > 0$.

In the case (i), there exist positive solutions x of (2.12) which verify (2.15).

Let x be a positive solution of (2.12) defined in $[t_1,+\infty)$, where $t_1 \geq t_0$. By two integrations, we get that

$$x'(t) \leq c_1 + \int_{t_1}^{t} |a(s)| w\left(\frac{x(s)}{s}\right) ds, \quad c_1 = |x'(t_1)|,$$

and

$$x(t) \leq c_2 + c_1(t-t_1) + \int_{t_1}^{t} (t-s)|a(s)| w\left(\frac{x(s)}{s}\right) ds, \quad c_2 = x(t_1) > 0.$$

The latter estimate leads to — recall that $t_0 \geq 1$ —

$$\frac{x(t)}{t} \leq c_1 + c_2 + \int_{t_1}^{t} |a(s)| w\left(\frac{x(s)}{s}\right) ds, \quad t \geq t_1. \tag{2.16}$$

According to Bihari's direct inequality (1.1), (1.3) for $c = c_1 + c_2$, $g = w$ and $k = 1$, we arrive at

$$\frac{x(t)}{t} \leq G^{-1}\left(G(c) + \int_{t_1}^{+\infty} |a(s)| ds\right) = X = X(c,t_1) < +\infty, \quad t \geq t_1.$$

Now, notice that

$$\begin{aligned} 0 \leq \int_{t_1}^{t} \left| a(s) w\left(\frac{x(s)}{s}\right)\right| ds &\leq \int_{t_1}^{t} |a(s)| \cdot w(X)\, ds \\ &\leq w(X) \cdot \int_{t_1}^{+\infty} |a(s)| ds. \end{aligned} \tag{2.17}$$

In particular, we deduce that the integral

$$\int_{t_1}^{+\infty} a(s) w\left(\frac{x(s)}{s}\right) ds$$

is convergent.

Consequently, we have

$$\lim_{t\to+\infty} x'(t) = \lim_{t\to+\infty}\left[x'(t_1)+\int_{t_1}^{t} a(s)w\left(\frac{x(s)}{s}\right)ds\right]$$
$$= x'(t_1)+\int_{t_1}^{+\infty} a(s)w\left(\frac{x(s)}{s}\right)ds = l_x. \qquad (2.18)$$

The remaining part of (2.15) follows by means of L'Hôpital's rule.

Further, fix $c_1 = x'(t_1) = c_2 = d > 0$ and take $t_1 > t_0$ large enough to have

$$d > w(X)\cdot\int_{t_1}^{+\infty}|a(s)|ds.$$

This is always possible since

$$\lim_{t_1\to+\infty}\frac{d}{w(X(2d,t_1))} = \frac{d}{w(2d)} > 0 = \lim_{t_1\to+\infty}\int_{t_1}^{+\infty}|a(s)|ds.$$

By taking into account (2.18), we deduce that

$$l_x \geq d - \int_{t_1}^{+\infty}|a(s)|\cdot w(X)\,ds > 0.$$

In conclusion, there exist positive solutions x of (2.12) with $l_x > 0$.

Moving to the case (i), set $k > 0$ small enough to have

$$\int_{t_1}^{+\infty}|a(s)|ds < \int_{2k}^{1}\frac{du}{g(u)} < +\infty.$$

Fix also $c_1 = x'(t_1) > 0$, $c_2 > 0$ such that $c = c_1 + c_2 \in (k, 2k)$. We have

$$G(c)+\int_{t_1}^{+\infty}|a(s)|ds < G(2k)+\int_{2k}^{1}\frac{du}{g(u)} = G(1)$$

and so the inequality (2.16) implies that

$$\frac{x(t)}{t} \leq X < 1, \quad t \geq t_1.$$

The conclusion is reached in the same way as before.

In a slightly different formulation, the result presented in this subsection is due to Bihari [3, pp. 277–278].

In [8, pp. 360–361] several examples are given of equations (2.12) in the case (i) which posses solutions x not satisfying (2.15). The existence of positive solutions x with $\liminf_{t\to+\infty} x(t) > 0$ and $l_x = 0$ of equation (2.12) is discussed in [9, p. 159].

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