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Free Style Problem Solver

Second order linear inhomogeneous ODEs

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To Lidia Aştefanei †

Disclaimer

This essay has not been submitted to a referee. Therefore its content must be taken "as is."

The author welcomes your comments to his e-mail address[1] and thanks you in advance for your effort.

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Craiova, May 18, 2015 *O.G.M.*

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Preface

Many undergraduate problems in mathematics and mechanics reduce to finding solutions to (in)homogeneous second order ordinary differential equations (aka ODEs).

A set of computations, based on the simple *Bellman's estimate*, is always useful in this respect. In the author's experience, it is a pretty good time saver.

Craiova, [December 15, 2017] *O.G.M.*

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Acronyms

ODE Ordinary differential equation

Chapter 1
Bellman's estimate

1.1 An integral inequality

Given the interval $I \subset \mathbb{R}$, with $[t_0, T] \subseteq I$, consider the next inequality

$$x(t) \leq x_0 + \int_{t_0}^{t} a(s)x(s)ds, \qquad t \in [t_0, T], \tag{1.1}$$

where $x_0 \geq 0$ is a constant and a, x are continuous, non–negative valued functions defined in I.

Set $\varepsilon > 0$. The integral inequality still holds if we replace x_0 with $x_0 + \varepsilon$. The difference is that now the right–hand member of the inequality is a positive quantity and we can write

$$\frac{a(t)x(t)}{x_0 + \varepsilon + \int_{t_0}^{t} a(s)x(s)ds} \leq a(t), \qquad t \in [t_0, T]. \tag{1.2}$$

Since

$$\frac{d}{dt}\left[\ln\left(x_0 + \varepsilon + \int_{t_0}^{t} a(s)x(s)ds\right)\right] = \frac{a(t)x(t)}{x_0 + \varepsilon + \int_{t_0}^{t} a(s)x(s)ds},$$

an integration of (1.2) with respect to t over $[t_0, T]$ leads to

$$\ln\left(\frac{x_0 + \varepsilon + \int_{t_0}^{t} a(s)x(s)ds}{x_0 + \varepsilon}\right) \leq \int_{t_0}^{t} a(s)ds.$$

Now,

$$x_0 + \varepsilon + \int_{t_0}^{t} a(s)x(s)ds \leq (x_0 + \varepsilon)\mathrm{e}^{\int_{t_0}^{t} a(s)ds}, \qquad t \in [t_0, T].$$

Taking into account (1.1), we get

$$x(t) \le (x_0 + \varepsilon) e^{\int_{t_0}^{t} a(s)ds}, \qquad t \in [t_0, T].$$

Finally, by making $\varepsilon \searrow 0$, we obtain

$$x(t) \le x_0 e^{\int_{t_0}^{t} a(s)ds}, \qquad t \in [t_0, T]. \tag{1.3}$$

1.2 Bellman's estimate

Consider the second order linear homogeneous ODE below

$$y'' + a(t)y' + b(t)y = 0, \qquad t \in [t_0, T], \tag{1.4}$$

where the functions a, b are defined in I and continuous. The solution y is always assumed C^2, that is twice continuously differentiable.

The equation (1.4) can be rewritten as a system of two first order ODEs by introducing the new variable $v = y'$. We have

$$\begin{cases} y' = v, \\ v' = -b(t)y - a(t)v, \end{cases} \qquad t \in [t_0, T].$$

The initial data for the equation (1.4) read as

$$y(t_0) = y_0, \quad y'(t_0) = v(t_0) = y_1$$

for $y_0, y_1 \in \mathbb{R}$.

An integration with respect to t over $[t_0, T]$ leads to

$$y(t) - y_0 = \int_{t_0}^{t} v(s)ds$$

and

$$v(t) - y_1 = -\int_{t_0}^{t} [b(s)y(s) + a(s)v(s)]ds.$$

The triangle inequality helps us estimate that

$$|y(t)| \le |y_0| + \int_{t_0}^{t} |v(s)|ds \tag{1.5}$$

and

$$|v(t)| \le |y_1| + \int_{t_0}^{t} [|b(s)| \cdot |y(s)| + |a(s)| \cdot |v(s)|]ds. \tag{1.6}$$

The sum of (1.5) and (1.6) yields

$$|y(t)| + |v(t)| \leq |y_0| + |y_1| + \int_{t_0}^{t} (1 + |a(s)| + |b(s)|) \cdot (|y(s)| + |v(s)|) ds$$

for every $t \in [t_0, T]$.

An application of formula (1.3), with

$$x(t) = |y(t)| + |v(t)|, \qquad x_0 = x(t_0),$$

leads to the simple estimate

$$|y(t)| + |y'(t)| \leq (|y_0| + |y_1|) \cdot \mathrm{e}^{\int_{t_0}^{t} (1+|a(s)|+|b(s)|) ds}, \tag{1.7}$$

known as *Bellman's estimate*. See also [1].

Chapter 2
Changing the variables

2.1 The case when $a \in C^1$

Consider now the second order inhomogeneous ODE

$$y'' + a(t)y' + b(t)y = f(t), \qquad t \in [t_0, T], \tag{2.1}$$

where the functions $a, b, f : I \to \mathbb{R}$ are continuous and, as before, $y \in C^2$.

Our aim in this section is to simplify this equation by reducing it to the formula

$$\frac{d^2z}{ds^2} + c(s)z = g(s), \qquad s \in [s_0, S], \tag{2.2}$$

where $J \subset \mathbb{R}$ is an interval, $[s_0, S] \subseteq J$ and c, g are continuous in J.

To this end, let us assume first that $a \in C^1$. We look for a change of variables

$$y = v(t)z$$

that will make the term "$a(t)y'$ " from (2.1) vanish.

The formulas

$$y' = v'z + vz', \quad y'' = v''z + 2v'z' + vz'',$$

once introduced into (2.1), yield

$$vz'' + [2v' + a(t)v]z' + [b(t)v + a(t)v' + v'']z = f(t), \qquad t \in [t_0, T].$$

We notice that, if $2v' + a(t)v = 0$, we can get rid of z' in the preceding equation. In this way, by taking

$$v(t) = \mathrm{e}^{-\frac{1}{2}\int a(t)dt} = \mathrm{e}^{-\frac{1}{2}\int_{t_0}^{t} a(s)ds},$$

we reduce the equation (2.1) to the simpler form (2.2), that is

$$z'' + c(t)z = g(t), \qquad t \in [t_0, T],$$

where

$$c(t) = \frac{b(t)v + a(t)v' + v''}{v(t)} = b(t) - \frac{[a(t)]^2}{4} - \frac{a'(t)}{2}$$

and

$$g(t) = \frac{f(t)}{v(t)} = f(t) \cdot \mathrm{e}^{\frac{1}{2}\int_{t_0}^{t} a(s)ds}.$$

2.2 The general case*

Let us notice that, by multiplying the equation (2.1) with

$$p(t) = \mathrm{e}^{\int a(t)dt} = \mathrm{e}^{\int_{t_0}^{t} a(s)ds},$$

we can recast it as

$$[p(t)y']' + p(t)b(t)y = p(t)f(t), \qquad t \in [t_0, T].$$

We would like to have

$$p(t)y' = \frac{dy}{\frac{dt}{p(t)}} = \frac{dz}{ds},$$

which means that we are interested in finding $s = s(t)$ such that

$$z(s(t)) = y(t) \quad \text{and} \quad \frac{dz}{ds}(s(t)) = p(t)y'(t), \quad t \in [t_0, T]. \tag{2.3}$$

A differentiation of the first identity in (2.3) with respect to t yields

$$\frac{d}{dt}[z(s(t))] = \frac{dz}{ds}(s(t)) \cdot s'(t) = y'(t)$$

and, taking into account the second identity in (2.3), we obtain the initial value problem

$$\frac{ds}{dt} = \frac{1}{p(t)}, \qquad s(t_0) = s_0. \tag{2.4}$$

An integration with respect to t in $[t_0, T]$ leads to

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* May be omitted at first reading.

$$s(t) = s_0 + \int_{t_0}^{t} \frac{d\tau}{p(\tau)} = s_0 + \int_{t_0}^{t} \mathrm{e}^{-\int_{t_0}^{\tau} a(\xi)d\xi} d\tau.$$

Further,

$$[p(t)y']' = \frac{d}{dt}\left[\frac{dz}{ds}(s(t))\right] = \frac{d^2z}{ds^2} \cdot \frac{ds}{dt} = \frac{1}{p(t)} \cdot \frac{d^2z}{ds^2}.$$

The equation (2.1) reads now as

$$\frac{d^2z}{ds^2} + [p(t)]^2 b(t)z = [p(t)]^2 f(t), \qquad t \in [t_0, T]. \tag{2.5}$$

The inverse of the function $s = s(t)$, that is $t = t(s)$, verifies the initial value problem — recall (2.4) —

$$\frac{dt}{ds} = p(t), \qquad t(s_0) = t_0.$$

In conclusion, we get from (2.5) that

$$\frac{dz^2}{ds^2} + c(s)z = g(s), \qquad s \in [s_0, S],$$

where

$$c(s) = [p(t(s))]^2 b(t(s)), \qquad g(s) = [p(t(s))]^2 f(t(s)).$$

The general case is, obviously, more difficult than the first case discussed here. Fortunately, in most problems we encounter a continuously differentiable coefficient a. See also [3].

Chapter 3
The solution

3.1 The uniqueness

We are interested in this essay in solving the initial value problem

$$\begin{cases} y'' + a(t)y' + b(t)y = f(t), & t \in [t_0, T], \\ y(t_0) = y_0, \\ y'(t_0) = y_1. \end{cases} \tag{3.1}$$

According to the previous section, we first simplify the formula of the equation. Since in most undergraduate applications the continuous coefficients a, b are constant — meaning that a is always C^1 —, we shall keep the notation t for the argument of the unknown function z.

Before writing down the initial value problem for the simplified equation, let us notice that, given the change of variables

$$y(t) = \mathrm{e}^{-\frac{1}{2}\int_{t_0}^{t} a(s)ds} z(t), \qquad t \in [t_0, T], \tag{3.2}$$

we have

$$y'(t) = -\frac{1}{2}a(t) \cdot y(t) + \mathrm{e}^{-\frac{1}{2}\int_{t_0}^{t} a(s)ds} \cdot z'(t). \tag{3.3}$$

The new initial value problem reads as

$$\begin{cases} z'' + c(t)z = g(t), & t \in [t_0, T], \\ z(t_0) = z_0, \\ z'(t_0) = z_1, \end{cases} \tag{3.4}$$

where, taking into account (3.2), (3.3) for $t = t_0$, we have

$$\begin{cases} z_0 = y_0, \\ z_1 = \frac{a(t_0)}{2} \cdot y_0 + y_1. \end{cases}$$

Let us establish now that *the initial value problem (3.4) has a unique solution.* To this end, assume that z_1, z_2 verify both the problem (3.4).

Their difference, namely $Y = z_1 - z_2$, verifies the next initial value problem

$$\begin{cases} Y'' + c(t)Y = 0, & t \in [t_0, T], \\ Y(t_0) = 0, \\ Y'(t_0) = 0. \end{cases}$$

Now, according to Bellman's estimate (1.7), we deduce that

$$|Y(t)| + |Y'(t)| \leq (|Y(t_0)| + |Y'(t_0)|) \cdot \mathrm{e}^{\int_{t_0}^{t}(1+|c(s)|)ds} = 0, \quad t \in [t_0, T].$$

In conclusion, $z_1 = z_2$ throughout $[t_0, T]$.

3.2 The wronskian

Given the linear homogeneous ODE

$$y'' + a(t)y' + b(t)y = 0, \qquad t \in [t_0, T], \tag{3.5}$$

assume that we know a *particular solution* y_1 which is positive everywhere in $[t_0, T]$.

The *wronskian* of the pair (y_1, y) of solutions — here, y stands for the general solution of (3.5) — is defined by the formula

$$W[y_1, y](t) = \begin{vmatrix} y_1(t) & y(t) \\ y_1'(t) & y'(t) \end{vmatrix}, \qquad t \in [t_0, T].$$

We make the following computations

$$\begin{aligned} \frac{dW[y_1, y]}{dt} &= \begin{vmatrix} y_1'(t) & y'(t) \\ y_1'(t) & y'(t) \end{vmatrix} + \begin{vmatrix} y_1(t) & y(t) \\ y_1''(t) & y''(t) \end{vmatrix} = \begin{vmatrix} y_1(t) & y(t) \\ y_1''(t) & y''(t) \end{vmatrix} \\ &= \begin{vmatrix} y_1(t) & y(t) \\ -a(t)y_1'(t) - b(t)y_1(t) & -a(t)y'(t) - b(t)y(t) \end{vmatrix}. \end{aligned}$$

Recalling that, if we add the first row multiplied by "$b(t)$" to the second row, the determinant does not change, we get

$$\frac{dW[y_1, y]}{dt} = \begin{vmatrix} y_1(t) & y(t) \\ -a(t)y_1'(t) & -a(t)y(t) \end{vmatrix} = -a(t) \cdot W[y_1, y], \qquad t \in [t_0, T].$$

In conclusion, the wronskian verifies the formula

$$W[y_1, y](t) = C \cdot \mathrm{e}^{-\int a(t)dt} = W[y_1, y](t_0) \cdot \mathrm{e}^{-\int_{t_0}^{t} a(s)ds} \tag{3.6}$$

throughout $[t_0, T]$.

3.3 The general solution of the simplified equation

Given the simplified ODE below

$$z'' + c(t)z = 0, \qquad t \in [t_0, T],$$

assume that it has a positive valued solution z_1.

The wronskian reads in this case as

$$W[z_1, z](t) = \begin{vmatrix} z_1(t) & z(t) \\ z_1'(t) & z'(t) \end{vmatrix} = C, \qquad C \in \mathbb{R}.$$

Recasting this identity as a first order inhomogeneous ODE with respect to z, we obtain

$$z' = \frac{z_1'(t)}{z_1(t)} \cdot z + \frac{C}{z_1(t)}. \tag{3.7}$$

The general solution of the equation (3.7) reads as

$$\begin{aligned} z(t) &= C(t) \cdot \mathrm{e}^{\int \frac{z_1'(t)}{z_1(t)} dt} = C(t) \cdot \mathrm{e}^{\int_{t_0}^{t} \frac{z_1'(\tau)}{z_1(\tau)} d\tau} \\ &= C(t) z_1(t), \qquad t \in [t_0, T], \end{aligned}$$

where C is an yet unknown smooth function.

To find C, we introduce the preceding formula into the equation and get

$$C'(t) = \frac{C}{[z_1(t)]^2}$$

everywhere in $[t_0, T]$.

By an integration with respect to t, we conclude that the general solution of the simplified ODE reads as

$$\begin{aligned} z(t) &= C(t) z_1(t) = \int \frac{C}{[z_1(t)]^2} dt \cdot z_1(t) \\ &= C_1 \cdot z_1(t) + C_2 \cdot z_1(t) \int_{t_0}^{t} \frac{d\tau}{[z_1(\tau)]^2}, \end{aligned}$$

where the numbers $C_1, C_2 \in \mathbb{R}$ are the integration constants.

3.4 A particular solution of the inhomogeneous ordinary differential equation

Assuming that the homogeneous ODE

$$z'' + c(t)z = 0, \qquad t \in [t_0, T], \tag{3.8}$$

has a positive valued particular solution z_1, we would like to find *a particular solution z_p* of the inhomogeneous ODE

$$z'' + c(t)z = g(t), \qquad t \in [t_0, T]. \tag{3.9}$$

Let us try the following formula

$$z_p = D(t)z_1, \tag{3.10}$$

where D is an yet unknown smooth function with real values.

Inserting (3.10) into the equation (3.9), we get

$$z_1(t)D'' + 2z_1'(t)D' + [z_1''(t) + c(t)z_1(t)]D = g(t), \qquad t \in [t_0, T].$$

Notice that the sum between the brackets is zero, z_1 being a solution of the equation (3.8).

Introduce the function $E = E(t)$ with the formula $E = D'$. We can now recast the latter formula as an inhomogeneous first order ODE, that is

$$E' = -2\frac{z_1'(t)}{z_1(t)} \cdot E + \frac{g(t)}{z_1(t)}, \qquad t \in [t_0, T]. \tag{3.11}$$

As before, the general solution of this equation reads as

$$E(t) = C(t) \cdot \mathrm{e}^{-2\int \frac{z_1'(t)}{z_1(t)}dt} = \frac{C(t)}{[z_1(t)]^2} \tag{3.12}$$

for every $t \in [t_0, T]$, where C is an yet unknown smooth function.

The formula (3.12), once inserted into (3.11), yields

$$C'(t) = z_1(t)g(t).$$

Since we are looking for a particular solution of (3.11), we take

$$C(t) = \int_{t_0}^{t} z_1(s)g(s)ds.$$

Now, because of

$$D'(t) = \frac{C(t)}{[z_1(t)]^2} = \frac{1}{[z_1(t)]^2}\int_{t_0}^{t} z_1(s)g(s)ds, \quad t \in [t_0, T], \tag{3.13}$$

a variant of D is given by

$$D(t) = \int_{t_0}^{t} \frac{1}{[z_1(s)]^2}\int_{t_0}^{s} z_1(\tau)g(\tau)d\tau ds. \tag{3.14}$$

Via an integration by parts, we have that

$$\begin{aligned} &D(t)\\ &= \int_{t_0}^{t} \frac{d}{ds}\left\{\int_{t_0}^{s} \frac{d\tau}{[z_1(\tau)]^2}\right\} \cdot \left[\int_{t_0}^{s} z_1(\tau)g(\tau)d\tau\right] ds\\ &= \int_{t_0}^{t} \frac{d\tau}{[z_1(\tau)]^2} \cdot \int_{t_0}^{t} z_1(\tau)g(\tau)d\tau - \int_{t_0}^{t} \left\{z_1(s)\int_{t_0}^{s} \frac{d\tau}{[z_1(\tau)]^2}\right\} g(s)ds. \end{aligned}$$

Notice that, for our choice of D, — recall (3.13), (3.14) for $t = t_0$ —

$$D(t_0) = D'(t_0) = 0. \tag{3.15}$$

Finally, collecting all the details into (3.10), we obtain

$$\begin{aligned} z_p(t) &= z_1(t)\int_{t_0}^{t} \frac{d\tau}{[z_1(\tau)]^2} \cdot \int_{t_0}^{t} z_1(s)g(s)d\tau\\ &\quad - z_1(t)\int_{t_0}^{t} z_1(s)\int_{t_0}^{s} \frac{d\tau}{[z_1(\tau)]^2} \cdot g(s)ds\\ &= z_1(t)\int_{t_0}^{t} z_1(s)\left\{\int_{t_0}^{t} \frac{d\tau}{[z_1(\tau)]^2} - \int_{t_0}^{s} \frac{d\tau}{[z_1(\tau)]^2}\right\} g(s)ds\\ &= z_1(t)\int_{t_0}^{t} \left\{z_1(s)\int_{s}^{t} \frac{d\tau}{[z_1(\tau)]^2}\right\} g(s)ds, \qquad t \in [t_0, T]. \end{aligned} \tag{3.16}$$

Taking into account (3.15), we have also that

$$z_p(t_0) = D(t_0)z_1(t_0) = 0 \tag{3.17}$$

and

$$z_p'(t_0) = D'(t_0)z_1(t_0) + D(t_0)z_1'(t_0) = 0. \tag{3.18}$$

3.5 The solution of (3.4)

Since the problem (3.4) has a unique solution, we look for constants $C_1, C_2 \in \mathbb{R}$ so that the solution can be written as

$$\begin{aligned} z(t) &= \text{a particular solution of the equation (3.8)} + z_p(t)\\ &= C_1 z_1(t) + C_2 z_1(t)\int_{t_0}^{t} \frac{ds}{[z_1(s)]^2} + z_p(t), \qquad t \in [t_0, T]. \end{aligned}$$

The constants C_1, C_2 are found by means of the data from problem (3.4). To this end, we have — recall (3.17), (3.18) —

$$z(t_0) = C_1 z_1(t_0) + z_p(t_0) = z_0$$

and

$$z_1'(t_0) = C_1 z_1'(t_0) + C_2 \cdot \frac{1}{z_1(t_0)} + z_p'(t_0) = z_1.$$

We have obtained

$$C_1 = \frac{z_0}{z_1(t_0)}, \qquad C_2 = z_1 \cdot z_1(t_0) - z_0 \cdot z_1'(t_0).$$

Chapter 4
Particular cases

4.1 Constant coefficients

If the coefficients a, b of the general linear inhomogeneous ODE (1.4) are constant then the coefficient c of the simplified ODE (3.8) is also a constant,

$$c(t) = b - \frac{a^2}{4} = c.$$

We recall the fundamental formula (3.16),

$$z_p(t) = \int_{t_0}^{t} G(t,s)g(s)ds, \qquad G(t,s) = z_1(t)z_1(s)\int_s^t \frac{d\tau}{[z_1(\tau)]^2},$$

where $t_0 \le s \le t \le T$.

We have three cases. In the first one, $c = -\omega^2 < 0$ for some constant $\omega > 0$. We take $z_1(t) = \mathrm{e}^{\omega t}$. Here, the quantity $G(t,s)$ reads as

$$G(t,s) = \mathrm{e}^{\omega(t+s)} \int_s^t \frac{d\tau}{\mathrm{e}^{2\omega\tau}} = \frac{1}{\omega} \cdot \sinh \omega(t-s).$$

The solution of (3.4) is given by

$$z(t) = z_0 \cosh \omega(t-t_0) + \frac{z_1}{\omega} \sinh \omega(t-t_0) + \frac{1}{\omega}\int_{t_0}^{t} \sinh \omega(t-s) \cdot g(s)ds.$$

In the second case, $c = 0$. We take $z_1(t) = 1$. Here, the quantity $G(t,s)$ reads as

$$G(t,s) = t - s.$$

The solution of (3.4) is given by

$$z(t) = z_0 + z_1(t-t_0) + \int_{t_0}^{t} (t-s)g(s)ds.$$

In the third case, $c = \omega^2$ for some constant $\omega > 0$. We take $z_1(t) = \cos\omega t$ — the interval $[t_0, T]$ must be chosen appropriately! —. Here, the quantity $G(t,s)$ reads as

$$G(t,s) = \sin\omega t \sin\omega s \int_s^t \frac{d\tau}{\cos^2\omega\tau} = \frac{1}{\omega} \cdot \sin\omega(t-s).$$

The solution of (3.4) is given by

$$z(t) = z_0 \cos\omega(t-t_0) + \frac{z_1}{\omega}\sin\omega(t-t_0) + \frac{1}{\omega}\int_{t_0}^t \sin\omega(t-s)\cdot g(s)ds.$$

4.2 Perturbations

In many undergraduate textbooks, e.g. [2, pages 168, 176], one can find presentations of the so-called method of undetermined coefficients as some sort of independent enterprise and not as of a less-obvious application of the fundamental variation of parameters procedure. The computations in this section show that the method of undetermined coefficients is nothing but a disguised particular case of the method of variation of parameters.

Given the simplified inhomogeneous equation (3.9), assume that

$$c(t) = \pm\omega^2, \qquad g(t) = t^n e^{\alpha t}\cos\beta t, \tag{4.1}$$

where $\omega > 0$, $n \geq 1$ is an integer and $\alpha, \beta \in \mathbb{R}$.

The particular solution z_p can be computed in this case by taking into account formula (3.14), that is

$$z_p(t) = z_1(t)\int_{t_0}^t \frac{1}{[z_1(s)]^2}\int_{t_0}^s z_1(\tau)g(\tau)d\tau ds, \qquad t \in [t_0, T].$$

In fact, we shall work loosely and compute the solution without caring about the integration constants, namely

$$z_p(t) = z_1(t)\int \frac{1}{[z_1(t)]^2}\left[\int z_1(t)g(t)dt\right]dt,$$

since the integration constants will join the homogeneous part of the general solution (the expense will be on the initial data).

Now, given $\gamma \in \mathbb{C} - \{0\}$ and $m \geq 0$ an integer, the following formula

$$\int t^m e^{\gamma t} dt = e^{\gamma t}\left[\frac{1}{\gamma}\cdot t^m \right. \\ \left. + \ (\text{sign } m)\sum_{k=1}^m (-1)^k \frac{m(m-1)\cdots(m-k+1)}{\gamma^{k+1}}\cdot t^{m-k}\right] \tag{4.2}$$

can be established by mathematical induction.

Notice that the perturbation from (4.1) is the real part "*Re*" of the quantity

$$g(t) = t^n e^{\eta t}, \qquad \eta = \alpha + i\beta.$$

Take $z_1(t) = e^{-\lambda t}$, where $\lambda \in \{\omega, i\omega\}$. Then, taking into account (4.2) for $\gamma = \eta - \lambda \neq 0$, we obtain

$$\begin{aligned}
&\frac{1}{[z_1(t)]^2} \int z_1(t) g(t) dt \\
&= e^{(2\lambda+\gamma)t} \left[\frac{1}{\gamma} \cdot t^n + \sum_{k=1}^{n} (-1)^k \frac{n(n-1)\cdots(n-k+1)}{\gamma^{k+1}} \cdot t^{n-k} \right].
\end{aligned}$$

Further, by assuming that $2\lambda + \gamma = \eta + \lambda \neq 0$, we obtain

$$\begin{aligned}
&z_1(t) \int \frac{1}{[z_1(t)]^2} \left[\int z_1(t) g(t) dt \right] dt \\
&= e^{-\lambda t} \left[\int \frac{1}{\gamma} \cdot t^n e^{(2\lambda+\gamma)t} dt \right. \\
&\left. + \sum_{k=1}^{n} (-1)^k \frac{n(n-1)\cdots(n-k+1)}{\gamma^{k+1}} \int t^{n-k} e^{(2\lambda+\gamma)t} dt \right] \\
&= e^{-\lambda t} \left\{ \frac{e^{(2\lambda+\gamma)t}}{\gamma} \left[\frac{t^n}{2\lambda+\gamma} + \sum_{p=1}^{n} (-1)^p \frac{n(n-1)\cdots(n-p+1) t^{n-p}}{(2\lambda+\gamma)^{p+1}} \right] \right. \\
&+ \sum_{k=1}^{n-1} (-1)^k \frac{n(n-1)\cdots(n-k+1)}{\gamma^{k+1}} \\
&\times e^{(2\lambda+\gamma)t} \left[\frac{t^{n-k}}{2\lambda+\gamma} + \sum_{p=1}^{n-k} (-1)^p \frac{(n-k)\cdots(n-k-p+1) t^{n-k-p}}{(2\lambda+\gamma)^{p+1}} \right] \\
&\left. + (-1)^n \frac{n!}{\gamma^{n+1}} \cdot \frac{e^{(2\lambda+\gamma)t}}{2\lambda+\gamma} \right\} \\
&= e^{(\lambda+\gamma)t} \left\{ \frac{1}{\gamma(2\lambda+\gamma)} \cdot t^n \right. \\
&+ \sum_{p=1}^{n} (-1)^p n \cdots (n-p+1) \left[\frac{1}{\gamma(2\lambda+\gamma)^{p+1}} + \frac{1}{\gamma^{p+1}(2\lambda+\gamma)} \right] \cdot t^{n-p} \\
&\left. + \sum_{k=1}^{n-1} \sum_{p=1}^{n-k} (-1)^{k+p} \frac{n \cdots (n-k-p+1)}{\gamma^{k+1}(2\lambda+\gamma)^{p+1}} \cdot t^{n-k-p} \right\} \\
&= e^{\eta t} \cdot (n\text{–th order polynomial in } t).
\end{aligned}$$

If $\gamma = \eta - \lambda \neq 0$ and $\eta + \lambda = 0$ then

$$
\begin{aligned}
&z_1(t)\int \frac{1}{[z_1(t)]^2}\left[\int z_1(t)g(t)dt\right]dt\\
&= \mathrm{e}^{-\lambda t}\cdot \begin{cases} \frac{1}{2\gamma}\cdot t^2 - \frac{1}{\gamma^2}\cdot t, \quad \text{when } n=1,\\ \frac{1}{(n+1)\gamma}\cdot t^{n+1} + \sum\limits_{k=1}^{n}(-1)^k \frac{n\cdots(n-k+2)}{\gamma^{k+1}}\cdot t^{n-k+1}, \quad \text{when } n\geq 2,\end{cases}\\
&= \mathrm{e}^{-\lambda t} t\cdot \begin{cases} \frac{1}{2\gamma}\cdot t - \frac{1}{\gamma^2}, \quad \text{when } n=1,\\ \frac{1}{(n+1)\gamma}\cdot t^{n} + \sum\limits_{k=1}^{n}(-1)^k \frac{n\cdots(n-k+2)}{\gamma^{k+1}}\cdot t^{n-k}, \quad \text{when } n\geq 2,\end{cases}\\
&= t\mathrm{e}^{\eta t}\cdot (n\text{–th order polynomial in } t).
\end{aligned}
$$

If $\gamma = \eta - \lambda = 0$ then

$$
\begin{aligned}
&z_1(t)\int \frac{1}{[z_1(t)]^2}\left[\int z_1(t)g(t)dt\right]dt\\
&= \mathrm{e}^{\lambda t}\cdot \begin{cases} \frac{1}{4\lambda}\cdot t^2 - \frac{1}{4\lambda^2}\cdot t + \frac{1}{8\lambda^3}, \quad \text{when } n=1,\\ \frac{1}{(n+1)(2\lambda)}\cdot t^{n+1} + \sum\limits_{k=1}^{n+1}(-1)^k \frac{n\cdots(n-k+2)}{(2\lambda)^{k+1}}\cdot t^{n-k+1}, \quad \text{when } n\geq 2,\end{cases}\\
&= \mathrm{e}^{\lambda t}\cdot [(n+1)\text{–th order polynomial in } t]\\
&= t\mathrm{e}^{\eta t}\cdot (n\text{–th order polynomial in } t) + \text{constant}\cdot \mathrm{e}^{\lambda t}.
\end{aligned}
$$

Since the latter term of this sum is a solution of the homogeneous (part of the) equation, we can neglect it.

In all of these three situations, the particular solution reads as

$$
z_p(t) = Re\left\{z_1(t)\int \frac{1}{[z_1(t)]^2}\left[\int z_1(t)g(t)dt\right]dt\right\}, \quad t\in[t_0,T].
$$

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